Inequalities

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1 Standard Inequalities

- For any $x \in \mathbb{R}$, $x^2 \geq 0$ (equality when $x = 0$).
- AM-GM: If $x_1, \ldots, x_n$ are nonnegative reals, then

  $$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

  (equality when all the x_i are equal).
- Weighted AM-GM: If $x_1, \ldots, x_n$ are nonnegative reals and $w_1, \ldots, w_n$ are positive reals with sum 1, then

  $$w_1 x_1 + \cdots + w_n x_n \geq x_1^{w_1} \cdots x_n^{w_n}$$

  (equality when all the x_i are equal).

  (Weighted) AM-GM is perhaps the most generally useful inequality but requires some skill to use because expressions x_i and weights w_i must be chosen carefully. One guide for figuring this out is that the x_i must all be equal at the equality cases of the inequality that you are trying to prove.
- "Bunching": One important application of weighted AM-GM is to inequalities of symmetric polynomials. Given two sequences $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ satisfying
  - $a_1 + a_2 + \cdots + a_i \geq b_1 + b_2 + \cdots + b_i$ for any i
  - $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$,

  weighted AM-GM can be used to obtain

  $$\sum_{\text{sym}} x_1^{a_1} \cdots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} \cdots x_n^{b_n}$$

  for any nonnegative reals $x_1, \ldots, x_n$.
- Schur's inequality: If x, y, z are nonnegative reals and $r > 0$, then

  $$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0$$

  (equality when $x = y = z$ OR two of x, y, z are equal and the third is zero).
- Cauchy-Schwarz: For any real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$,

  $$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2$$

  (equality when the two sequences are proportional).

  Like weighted AM-GM, this inequality is very useful but is often tricky to figure out how to use. It is again useful to think about equality cases.

2 Other Techniques

- Smoothing: Suppose you are trying to prove a symmetric inequality of the form $f(x_1, \dots, x_n) \geq C$ subject to the constraint $x_1 + \cdots + x_n = ns$, and there is equality when $x_1 = \cdots = x_n = s$. Then one can try to "smooth" the x_i together by replacing two of the variables, say $x_1 < s$ and $x_2 > s$, with s and $x_1 + x_2 - s$. If one can show that
$$f(x_1, x_2, x_3, \dots, x_n) \geq f(s, x_1 + x_2 - s, x_3, \dots, x_n),$$
then by repeating this smoothing procedure one has a chain of inequalities
$$f(x_1, \dots, x_n) \geq \cdots \geq f(s, s, \dots, s) = C.$$
Variants: unsmoothing, linear functions achieve extremal values at endpoints.

- Substitutions: Finding a clever change of variables can simplify an inequality tremendously. Here are a few standard ones to keep in mind:
  - Trig substitutions: if you see something like $\sqrt{1 \pm x^2}$, substituting $x = \sin\theta$ or $x = \tan\theta$ is worth thinking about to eliminate the square root.
  - Sides of a triangle: Some three-variable inequalities are stated with the constraint that a, b, c are length of sides of a triangle. To eliminate this constraint, set $a = y + z, b = z + x, c = x + y$.
  - Cyclic substitution: In three variable inequalities, sometimes the change of variables $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$ is helpful. Note that this substitution doesn't always make sense!

3 Problems

1. Prove that for any positive reals a, b, c,
$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

2. Prove that for any nonnegative reals x, y, z,
$$\sqrt{3x^2 + xy} + \sqrt{3y^2 + yz} + \sqrt{3z^2 + zx} \leq 2(x + y + z).$$

3. (IMO 95/2) Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

4. (Bulgaria 95) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for all $i = 1, 2, \dots, n$. Show that
$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
and determine when there is equality.

5. Prove that for any $a, b, c, d \in \mathbb{R}$,
$$a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2+6abcd \geq a^2(bc+cd+db)+b^2(cd+da+ac)+c^2(da+ab+bd)+d^2(ab+bc+ca).$$
When does equality occur?

6. (USAMO 97/5) Prove that for any positive reals a, b, c,
$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$

7. Let a, b, c be side lengths of a triangle. Prove that

$$2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) \geq a^3 + b^3 + c^3 + 9abc.$$

8. Let $P(x)$ be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$.

9. Let a, b, c be positive reals with product 1. Show that

$$5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq (1+a)(1+b)(1+c).$$

10. (Iran 98) Let x, y, z be real numbers greater than 1 such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}$$

11. Prove that for any $a, b, c \in \mathbb{R}$,

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3 b + b^3 c + c^3 a).$$

12. (China TST 2005) Let $a, b, c, d > 0$ and $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

13. (IMO 92/5) Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the orthogonal projections of S onto the yz, zx, xy planes, respectively. Show that

$$|S|^2 \leq |S_x||S_y||S_z|.$$

14. (ISL 01/A3) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$